

# Corners in M-theory

Hisham Sati \*

Department of Mathematics  
University of Maryland  
College Park, MD 20742

## Abstract

M-theory can be defined on closed manifolds as well as on manifolds with boundary. As an extension, we show that manifolds with corners appear naturally in M-theory. We illustrate this with four situations: The lift to bounding twelve dimensions of M-theory on Anti de Sitter spaces, ten-dimensional heterotic string theory in relation to twelve dimensions, and the two M-branes within M-theory in the presence of a boundary. The M2-brane is taken with (or as) a boundary and the worldvolume of the M5-brane is viewed as a tubular neighborhood. We then concentrate on (variant) of the heterotic theory as a corner and explore analytical and geometric consequences. In particular, we formulate and study the phase of the partition function in this setting and identify the corrections due to the corner(s). The analysis involves considering M-theory on disconnected manifolds, and makes use of the extension of the Atiyah-Patodi-Singer index theorem to manifolds with corners and the  $b$ -calculus of Melrose.

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\*e-mail: [hsati@math.umd.edu](mailto:hsati@math.umd.edu)

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## 1 Introduction

The boundary of a manifold  $X$  cannot itself be a boundary, but there are situations in physics (and mathematics) which demand that sense be made out of some variation on the notion of boundaries of boundaries. For example, we know that heterotic string theory should be a boundary of M-theory [20] [21]. The latter in turn is best described globally using a bounding twelve-dimensional theory [48] [8]. A naive boundary of a boundary does not exist as e.g. the boundary operator in homology is nilpotent. However, this has a natural setting within manifolds with corners. With this we then can view the heterotic theory as codimension-two corner of the twelve-dimensional theory.

We recall the basics of manifolds with corners in section 2.1. Closed manifolds and manifolds with boundary are special cases of manifolds with corners. Also products of two manifolds with boundary form an interesting class of examples within manifolds with corners of codimension-two. We consider situations in M-theory, in addition to the heterotic boundary, where manifolds with corners are needed in order to describe the physical system. This is discussed in section 2.2 and includes the following.

**AdS/CFT correspondence.** Eleven-dimensional supergravity admits solutions with an Anti-de Sitter space as a factor, most famously  $\text{AdS}_4 \times S^7$  [16] and  $\text{AdS}_7 \times S^4$  [38]. These spaces appear as the near horizon limits of the M2-brane and the M5-brane, respectively, via the AdS/CFT correspondence which relates quantum supergravity on the AdS factor to the conformal field theory on the boundary  $\partial\text{AdS}$  [30]. In section 2.2.1 we show that the lift of these solutions to twelve dimensions leads naturally to a manifold with corners as the product of two manifolds with boundaries.

**The M5-brane.** The M5-brane with worldvolume  $W^6$  can be described in terms of tubular neighborhoods in the target 11-dimensional manifold  $Y^{11}$ . When  $Y^{11}$  has no boundary the resulting manifold arising from the sphere bundle of the normal bundle to the embedding  $W^6 \hookrightarrow Y^{11}$  has a boundary. Now M-theory itself on a manifold with boundary certainly makes sense [21] [7] [45] and so we ask what happens to the description of the M5-brane in that case. While the main theme in [7] [45] was for when  $Y^{11}$  has a boundary, the description of the M5-brane was given only for the case when  $Y^{11}$  is closed (in [7] the M5-brane was related to M-theory with boundary only via anomalies involving torsion). In section 2.2.2 we show how the resulting manifold will be a manifold with corners of codimension-two. This uses the general result of [23]

that the removal of a tubular neighborhood of any submanifold creates a manifold of one codimension higher. Therefore, including an M5-brane on an eleven-dimensional manifold with a boundary leads naturally to a manifold with corners.

**The M2-brane.** The M2-brane in M-theory can have boundaries on the M5-brane [46]. When M-theory is considered on a manifold with boundary, then the two-dimensional boundaries of the M2-brane can end on the ten-dimensional boundary of M-theory, in the Horava-Witten set-up [21]. Therefore, we take the M2-brane within M-theory with boundary, that is on a twelve manifold which is a product of a three-manifold with boundary with an eight manifold. Then we wrap the M2-brane on the former; the ten-manifold which is the product of the membrane boundary with the eight-manifold will be a corner for the twelve-manifold. In fact, this will be essentially the heterotic corner. We describe this in section 2.2.3.

**Heterotic M-theory.** Heterotic string theory is essentially a boundary of M-theory [20] [21]. The M-theory partition function on a Spin eleven-manifold with boundary was considered in [7] with an emphasis on eleven rather than on twelve-dimensions. For topological and global (e.g. index theory) purposes, M-theory in turn is considered as a boundary of the bounding twelve-dimensional theory on  $Z^{12}$  [48] [8]. Hence, in the connection to heterotic string theory, the bounding twelve-dimensional theory requires having seemingly a ‘boundary of a boundary’. In section 3 we consider the effect of studying this from the point of view of the bounding twelve-dimensional theory. We provide two formulations, one using Dirac operators (in section 3.1) and another involving the signature operators (in section 3.2), making use of the emergence of the latter in [45].

As in [14], we consider the Horava-Witten theory on a ten-dimensional Spin manifold  $M^{10}$  from two points of view. First, via the product with the interval  $[0, 1] \times M^{10}$  (the “upstairs” formulation), which for us nicely connects to manifolds with corners by taking a further Cartesian product with another interval. We do this for most of section 3. Second, via  $S^1/\mathbb{Z}_2 \times M^{10}$  (the “downstairs” formulation) with  $\mathbb{Z}_2$  acting as an orientation-reversing involution. We consider the eleven-manifold as a boundary of a twelve-manifold in the presence of an orientation-reversing involution and study the effect on the signature operator and the corresponding eta-invariants in section 3.2.2. This allows us to formulate the phase of the partition function.

We study analytical and geometric aspects of the theory in this setting using mainly the constructions in [3] [17] and [36] and the survey [28]. In particular we consider the global reduction to ten dimensions of the phase of the partition function, using  $b$ -eta-invariants within the  $b$ -calculus [33]. This allows for more general boundary conditions than those of Atiyah-Patodi-Singer (APS) [2] used in [8]. The discussion requires considering M-theory on disconnected eleven-dimensional spaces. We also consider the case of multiple ten-dimensional (heterotic) components in the setting of manifolds with corners.

While this is mostly a physics paper, we have chosen to identify main (physical) results and observations by recording them as propositions and lemmas, mainly as a way of keeping track of the main statements.

## 2 Manifolds with corners and their relevance in M-theory

We first recall in section 2.1 the basics of manifolds with corners and then we provide our applications to M-theory in section 2.2.

### 2.1 Basic definitions and relevant tools

We now give the basic definitions and some of the properties that we need in the applications to M-theory, which we discuss starting in the following section.

**The basic definitions.** A *differentiable manifold with corners* is a topological space covered by charts which are locally open subsets of  $\mathbb{R}_+^n = [0, \infty)^n$  [9] [5]. Adding information about faces leads to *manifold with faces*. Imposing conditions on how the faces piece globally together leads to a restrictive class called  $\langle n \rangle$ -manifolds [23]. This is a manifold with faces together with an ordered  $n$ -tuple  $(\partial_0 X, \partial_1 X, \dots, \partial_{n-1} X)$  of faces of  $X$  which satisfy the following conditions:

- (1) The boundary is formed of  $n$  disconnected components  $\partial X = \partial_0 X \cup \dots \cup \partial_{n-1} X$ ;
- (2) The intersection  $\partial_i X \cap \partial_j X$  is a face of  $\partial_i X$  and of  $\partial_j X$  for all  $i \neq j$ .

The number  $n$  is called the *codimension* of  $X$ . We will be mainly interested in the case  $n = 2$ .

**Products and codimension.** The product of an  $\langle m \rangle$ -manifold with an  $\langle n \rangle$ -manifold  $\langle m+n \rangle$ -manifold. A  $\langle 0 \rangle$ -manifold is a manifold without a boundary while a  $\langle 1 \rangle$ -manifold is a manifold with boundary. So we can create many manifolds with boundary by multiplying manifolds of these two different types. Furthermore, we can create  $\langle n \rangle$ -manifolds from products of  $\langle 0 \rangle$ -manifolds with  $\langle n \rangle$ -manifolds,  $\langle 1 \rangle$ -manifolds with  $\langle n-1 \rangle$ -manifolds, and so on. In the main case of interest, which is 12-manifolds with corners of codimension-two, we can construct many such spaces by taking a product of a  $k$ -dimensional  $\langle i \rangle$ -manifold with a  $(12-k)$ -dimensional  $\langle 2-i \rangle$ -manifold with  $i = 0, 1$ . More explicitly, we can take the product of a closed manifold with a manifold with corners as well as the product of two manifolds with boundary, the sum of whose dimensions is 12.

We consider two simple examples of manifolds with corners of codimension-two.

**Example 1. The positive quadrant.** Let  $\overline{\mathbb{R}_+^2}$  denote the closed positive quadrant of  $\mathbb{R}^2$ , that is  $\overline{\mathbb{R}_+^2} = \{(x^1, x^2) \in \mathbb{R}^2 : x^1 \geq 0, x^2 \geq 0\}$ . The boundary of  $\overline{\mathbb{R}_+^2}$  in  $\mathbb{R}^2$  is the set of points at which one or both coordinates vanish. The points in  $\overline{\mathbb{R}_+^2}$  at which both coordinates vanish are called its *corner points*. The boundary of a smooth manifold with corners is in general not a smooth manifold with corners. For example, the boundary of  $\overline{\mathbb{R}_+^2}$  is the union  $\partial \overline{\mathbb{R}_+^2} = H_1 \cup H_2$ , where  $H_i = \{(x^1, x^2) \in \overline{\mathbb{R}_+^2} : x^i = 0\}$ ,  $i = 1, 2$ , is a one-dimensional smooth manifold with boundary.

**Example 2. Lie groups with action of maximal torus.** Let  $G$  be  $SU(2)$  or  $SO(4)$ , the Lie groups of rank 2, and let  $T^2$  be the corresponding maximal torus. Then  $T^2$  acts on the product  $(\mathbb{D}^2)^2$  of 2 disks  $\mathbb{D}^2$  by complex multiplication. The resulting associated fiber bundle  $G \times_{T^2} (\mathbb{D}^2)^2$  is a  $\langle 2 \rangle$ -manifold. For  $SU(2)$  this is five-dimensional, while for  $SO(4)$  this is eight-dimensional. For more on such examples see [25].

We will be interested in integrating forms on manifolds with corners. Integration over the boundary amounts to integrating over the boundary components. We illustrate this with an example.

**Example 3. The square in  $\mathbb{R}^2$ .** The square is a manifold with corners of codimension-two. Its edges are boundary hypersurfaces and its corners are codimension-two faces. Let  $I \times I = [0, 1] \times [0, 1]$  be the unit square in  $\mathbb{R}^2$ , and suppose  $\omega$  is a smooth 1-form on the boundary  $\partial(I \times I)$ . Consider the maps  $F_i : I \rightarrow I \times I$  given by

$$F_1(t) = (t, 0), \quad F_2(t) = (1, t), \quad F_3(t) = (1-t, 1), \quad F_4(t) = (0, 1-t). \quad (2.1)$$

The four curve segments in the sequence traverse the boundary of  $I \times I$  in the counterclockwise direction. Then Stokes' theorem for a manifold with corners gives [26]  $\int_{\partial(I \times I)} \omega = \int_{F_1} \omega + \int_{F_2} \omega + \int_{F_3} \omega + \int_{F_4} \omega$ . Such integration over rectangles should be familiar from electromagnetism, although it is usually not cast in this language. One of the main advantages of using manifolds with corners is that, for example, the cube which is not a smooth manifold would be smooth as a manifold with corners.

We will also need to study differential forms and cohomology on manifolds with corners.

**$L^2$ -cohomology.** A manifold with corners can be viewed as a manifold with singularities. De Rham cohomology does not capture the information at the singularities or corners. To make up for this, one restricts to the subcomplex of square-integrable differential forms, which leads to  $L^2$ -cohomology. Let  $(Y, g_Y)$  be a Riemannian manifold and let  $\Omega^p = \Omega^p(Y)$  be the space of smooth  $p$ -forms and  $L^2 = L^2(Y)$  the  $L^2$  completion of  $\Omega^p$  with respect to the  $L^2$ -metric. The differential  $d$  is defined to be the exterior differential with the domain  $\text{dom}(d) = \{\omega \in \Omega_{(2)}^p : d\omega \in L^2(Y)\}$ , where  $\Omega_{(2)}^p = \Omega^p(Y) \cap L^2(Y)$  is the space of square-integrable smooth  $p$ -forms. The  $L^2$ -cohomology is then the cohomology of the cochain complex

$$0 \longrightarrow \Omega_{(2)}^0(Y) \xrightarrow{d} \Omega_{(2)}^1(Y) \xrightarrow{d} \Omega_{(2)}^2(Y) \xrightarrow{d} \Omega_{(2)}^3(Y) \xrightarrow{d} \dots, \quad (2.2)$$

that is,  $H_{(2)}^p(Y) = \ker d_i / \text{Im } d_{i-1}$ . The natural map  $H_{(2)}^p(Y) \rightarrow H^p(Y; \mathbb{R})$  via the usual de Rham cohomology is an isomorphism for  $Y$  a compact manifold with corners because the  $L^2$  condition is automatically satisfied for all smooth forms. For a nice exposition on this see [6]. Hodge theory for a manifold with corners is discussed in [39] [34].

**Smoothing corners.** Manifolds with corners are smooth in the sense of having charts locally as open subsets of  $\mathbb{R}_+^n$ . However, they look like they should be singular at the corner. What is the explanation to this? One thing one could do is smooth out the corner via a diffeomorphism, which is not an isometry. For instance, if the corner is that of a quadrant then one can replace rectangular coordinates with polar coordinates and provide a smoothing of the corner by considering only nonzero value of the radial coordinate. This is called total boundary blow-up [32] (see also [28]). In our context will be interested in manifolds of the form  $Z \cong [0, 1]_{s_1} \times [0, 1]_{s_2} \times M$ , where  $s_1$  and  $s_2$  are Cartesian coordinates on the two intervals. Near the corner  $M$ , introduce polar coordinates via  $s_1 = r \cos \theta$  and  $s_2 = r \sin \theta$  so that the totally blown-up space is  $Z_{\text{tb}} \cong [0, \varepsilon)_r \times [0, \pi/2]_\theta \times M$ , for  $\varepsilon > 0$ . We have diffeomorphism instead of isometry because intersections of hypersurfaces at  $M$  do not have to occur at right angles, but any angle in the plane can be related by a diffeomorphism to the standard upper right quadrant. We will have this blow-up implicitly in mind in dealing with manifolds with corners.

To study the phase of the partition function we need to consider Dirac operators and their corresponding eta-invariants.

**Continuous spectrum and the  $b$ -trace.** A Dirac operator on a manifold with corners has a continuous spectrum, and hence trying to define the eta invariant will involve infinite traces. The way around this is to use the  $b$ -trace within the  $b$ -calculus [33]. For our purposes, the main idea can be summarized as follows (see also the next example). The  $b$ -trace is defined in terms of the  $b$ -integral for an operator  $O$  (schematically) as

$$\text{bTr}(O) := \text{b} \int_Y \text{tr}(O). \quad (2.3)$$

Then the corresponding eta-invariant will be defined using this trace as [33]

$$\text{b}\eta(D) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{t}} \text{bTr}(De^{-tD^2}) dt. \quad (2.4)$$

Let us illustrate this with an example, which will be useful for us later.

**Example 4. The interval over a manifold.** Consider  $[0, 1]_s$ , the unit interval with coordinate  $s$ , which will be fibered over a manifold  $M$ . The function  $ds/s$  is not integrable over  $[0, 1]_s$ , so that the corresponding heat operator is not trace class. However, the function  $s^z$ ,  $\text{Re } z > 0$ , is integrable with respect to  $ds/s$  over  $[0, 1]_s$ . This suggests using an integral which corresponds to the usual integral when  $z$  is zero. As nicely illustrated in [28], let  $f \in C^\infty(Y)$  be a smooth function on a manifold  $Y$ . Then for all complex numbers  $z$  with  $\text{Re } z > 0$ , the integral  $F(z) = \int_Y s^z f dg$  exists and it extends from  $\text{Re } z > 0$  to define a

meromorphic function on all of  $\mathbb{C}$ . Note that  $s^z = e^{z \log s}$  is an entire function of  $z$  for  $s > 0$ . Thus,  $f$  can be assumed to be supported on the collar  $[0, 1]_s \times M$  of  $Y$ . Then  $F(z)$  is well-defined for  $\text{Re } z > 0$  since  $s^z f(s, m)$  is integrable with respect to the measure  $(ds/s)dh$  as long as  $\text{Re } z > 0$ . Here  $m$  is a point in  $M$  and  $dh$  is a measure on  $M$ . Now expand  $f(s, m)$  in Taylor series at  $s = 0$ :  $f(s, m) \sim \sum_{k=0}^{\infty} s^k f_k(m)$ . Since the integral  $\int_{[0,1] \times M} s^{z+k} f_k(m) \frac{ds}{s} dh$  is equal to  $\frac{1}{z+k} \int_M f_k(m) dh$  then the function  $F(z)$  extends from  $\text{Re } z > 0$  to be a meromorphic function on  $\mathbb{C}$  with only simple poles at  $z = \{0, -1, -2, \dots\}$  with residue at  $z = 0$  given by  $\int_M f_0(m) dh = \int_M f(0, m) dh$ . The  $b$ -integral of  $f$  is the regular value of  $F(z)$  at  $z = 0$ ,  $\text{b} \int_Y f dg = \text{Reg}_{z=0} F(z)$ , such that the residue of  $F(z)$  at  $z = 0$  is given by  $\text{Res}_{z=0} F(z) = \int_Y f(0, m) dh$ .

We will also be interested in considering the kernels of Dirac operators on manifolds with corners.

**Infinite-dimensional kernels.** The dimensions of the kernels are generically infinite so that the Dirac operator is not Fredholm in general. We will consider the effect of this in section 3.1. We now illustrate this in the simple example of the square. Consider the Cauchy-Riemann operator  $\partial_z = \partial_x + i\partial_y$  on the square  $[0, 1]_x \times [0, 1]_y$ . The manifold and hence the operator are of product type. Then the kernel  $\ker \partial_z$  is infinite-dimensional since this kernel consists of all holomorphic functions on the square.

**Compactification of manifolds with cylindrical ends to manifold with corners.** In order to deal with the non-Fredholm property of the Dirac operator on the manifold with corners  $Z$ , one has to introduce another manifold  $\widehat{Z}$  of the same dimension which is formed by attaching infinite cylinders to the collars of  $Z$ . This will be used in section 3.1. The manifold  $\widehat{Z}$  can be compactified by introducing the change of variables  $x_1 = e^{s_1}$  and  $x_2 = e^{s_2}$ . As  $s_i \rightarrow \infty$ ,  $x_i \rightarrow 0$  and so this change of variables compactifies  $\widehat{Z}$  to be the interior of a compact manifold with corners of codimension-two  $Z$ . The metric then transforms to the  $b$ -metric as

$$g^Z = ds_1^2 + ds_2^2 + g^M \rightsquigarrow \text{b} g^Z = \left( \frac{dx_1}{x_1} \right)^2 + \left( \frac{dx_2}{x_2} \right)^2 + g^M. \quad (2.5)$$

**The Maslov index.** When the twelve-manifold  $Z^{12}$  has no boundary, there are nice additivity properties, for example the Novikov additivity of the signature [37]. The Atiyah-Patodi-Singer index theorem [2] gives the index of the Dirac and signature operators on manifolds with boundary in terms of the  $\hat{A}$ -genus and the L-genus, respectively, and the defects given by the corresponding eta invariants on the boundary. In the case of corners, the signature is no longer additive, but there is a correction term in Wall's nonadditivity [47]. The signature defect is the Maslov index of certain Lagrangian subspaces related to the cohomology of the boundary  $Y^{11}$  and the corner. This, and the corresponding generalization using [3] [17] [36] and [28], will be discussed in section 3.1 for the Dirac operator and in section 3.2 for the signature operator.

## 2.2 Occurrence in M-theory

In this section we consider four situations, three of which are related to M-branes in M-theory, and one related to heterotic string theory, where manifolds with corners appear naturally. The first one is M-theory on AdS spaces, which are configurations that occur as near horizon limits of M-branes. The second one arises by considering the M5-brane as a tubular neighborhood in eleven-dimensional spacetime with boundary. The third arises when considering boundaries in relation to the M2-brane. This includes the M2-brane having a boundary (ending on the M5-brane) or the M2-brane itself being considered as a boundary for instance when studying its partition function. The fourth views (a variant of) heterotic string theory as a corner in the twelve-dimensional bounding theory.

### 2.2.1 M-theory on AdS spaces

Anti-de Sitter space is a Lorentzian space with boundary at spatial infinity. The Euclidean version is given by a hyperbolic space, with very interesting boundary structure at infinity. Therefore considering the boundary

of AdS space amounts, in an appropriate sense, to looking at the boundary of M-theory as  $\partial\text{AdS}_i \times S^{11-i}$  for  $i = 4, 7$ . Compactifying the Euclidean boundary gives a product of spheres. In particular, for the M5-brane this gives  $S^3 \times S^7$ . Now the internal spaces  $S^7$  and  $S^4$  are boundaries of the 8-disk  $\mathbb{D}^8$  and the 5-disk  $\mathbb{D}^5$ , respectively. M-theory itself can be viewed as a boundary in twelve dimensions, so that from the point of view of this bounding theory we have spaces of the form  $\text{AdS}_i \times \mathbb{D}^{12-i}$  for  $i = 4, 7$ . We notice that both factors in the product are manifolds with boundaries, and hence the product itself is a manifold with corners of codimension-two, i.e. is a  $\langle 2 \rangle$ -manifold.

The internal spheres in the products with AdS spaces can also be replaced by homogeneous spaces  $G/H$ , where  $G$  and  $H$  are Lie groups, with analogous near horizon structures [4]. In fact, general Einstein spaces  $M^{11-i}$ , for  $i = 4, 7$ , with Killing spinors – and hence are Spin – can be used as well (see [12] and references therein). Thus, in order to detect corners, we would like to ask whether the spaces  $\text{AdS}_i \times M^{11-i}$  can be lifted to twelve dimensions. This reduces to checking whether  $M^{11-i}$  can be boundaries. For  $M^7$  this is always the case since the relevant bordism group is trivial  $\Omega_7^{\text{Spin}} = 0$ ; that is the Spin manifold  $M^7$  is always the boundary of some eight-manifold, say  $W^8$ . However, for  $M^4$  this is not the case since the bordism group is not trivial,  $\Omega_4^{\text{Spin}} = \mathbb{Z}$ . By Rohlin's theorem, a closed oriented Spin 4-manifold  $M^4$  is null cobordant in  $\Omega_4^{\text{Spin}}$ , i.e. is the boundary of a compact oriented Spin smooth 5-manifold  $W^5$  if and only if the signature  $\sigma(M^4)$  of  $M^4$  vanishes. Thus the signature is a complete cobordism invariant. The isomorphism  $\Omega_4^{\text{Spin}} \xrightarrow{\cong} \mathbb{Z}$  sends any cobordism class  $[M^4]$  to  $\sigma(M^4)/16$ . In particular, the Kummer surface  $K_4 = \{z_1^4 + z_2^4 + z_3^4 + z_4^4\} \subset \mathbb{C}P^3$ , whose signature is  $\sigma(K_4) = -16$ , provides a generator for  $\Omega_4^{\text{Spin}}$ . We have

**Proposition 1** (i) *The near horizon limit of the M2-brane can always be described as corner for the twelve-dimensional bounding theory.*

(ii) *The near horizon limit of the M5-brane can be described as a corner for the twelve-dimensional bounding theory provided that the internal four-manifold is an Einstein space with zero signature.*

We will consider the M5-brane and the M2-brane themselves in section 2.2.2 and section 2.2.3, respectively.

Examples of Spin 4-manifolds with zero signature include the 4-sphere  $S^4$ , the projective space  $\mathbb{R}P^4$  and their quotients by finite groups. Dimensional reductions of the latter type are considered e.g. in [13]. Classes of examples include ones for which the  $\hat{A}$ -genus vanishes, since in four dimensions the  $\hat{A}$ -genus and the Hirzebruch  $L$ -genus are related by a simple numerical factor. By the result of Atiyah-Hirzebruch the  $\hat{A}$ -genus vanishes if the manifold admits a smooth (isometric) circle action [1]. Interestingly, even in the non-Spin case (say for us  $\text{Spin}^c$ ), such a result still holds [18]. The resulting theory on the orbit of the circle action is ten-dimensional type IIA string theory. The M5-brane will give rise to a type IIA NS5-brane, which is of the same dimension, so that the dimension of the transverse space is reduced by one. Therefore, we have the nice compatibility result

**Proposition 2** *The near horizon limit of the M5-brane can be described as a corner when M-theory is taken with a circle action, that is when the theory is related to type IIA string theory.*

For example, for  $M^4 = S^3 \times S^1$  this leads to type IIA string theory on  $\text{AdS}_7 \times S^3$ , studied e.g. in [41]. On the other hand, for  $S^7$  the circle action gives a supersymmetric background in type IIA string theory of the form  $\text{AdS}_4 \times \mathbb{C}P^3$  first considered in [10]. In these cases the ten-dimensional corners are  $\partial\text{AdS}_7 \times S^3 \times S^1$  and  $\partial\text{AdS}_4 \times S^7$ , respectively.

## 2.2.2 The M5-brane as a tubular neighborhood

Here we consider the extension of the description of the M5-brane as a tubular neighborhood to the case when  $Y^{11}$  has a boundary. This results, upon removing of a tubular neighborhood, in a manifold with corners of codimension-two.

Consider an M5-brane with worldvolume  $W^6$ , considered as a (closed) submanifold inside a closed eleven-manifold  $Y^{11}$ . Removing a tubular neighborhood of the M5-brane leads to a manifold with a boundary, as illustrated in [7] and used in [45]. While both of these references are concerned mainly with the case when  $Y^{11}$  has a boundary, that was restricted to a closed  $Y^{11}$  when dealing with tubular neighborhoods. Now we provide a description of the case when  $\partial Y^{11} \neq \emptyset$  using the formalism in [5] [9] [23].

Let  $\iota : W^6 \hookrightarrow Y^{11}$  be the embedding of the M5-brane in spacetime with normal bundle  $N^{11} \rightarrow W^6$ , viewed as a tubular neighborhood of  $W^6$  in  $Y^{11}$ . The unit sphere bundle of radius  $r$  is the associated bundle  $S^4 \rightarrow \mathcal{S}^{10} \rightarrow W^6$ , and the corresponding disk bundle of radius  $r$  is  $\mathbb{D}^5 \rightarrow \mathcal{D}^{11} \rightarrow W^6$ . Removing this disk bundle leads to an eleven-manifold  $Y_r^{11} = Y^{11} - \mathcal{D}^{11}$  with boundary  $\partial Y_r^{11} = \mathcal{S}^{10}$ , the sphere bundle.

If  $Y^{11}$  is a manifold with boundary then the removal of a tubular neighborhood of the M5-brane from  $Y^{11}$  will result in a manifold with corners of codimension-two. Then, assuming that  $Y^{11}$  has multiple boundary components  $\partial_i Y^{11}$  ( $i = 1, \dots, n$ ),  $W^6$  is a manifold with faces and becomes a manifold with boundary if we identify  $\partial_i W^6 = W^6 \cap \partial_i Y^{11}$ . We can interpret this as the boundary of M5-brane on the M9-brane, or the M5-brane in heterotic M-theory.

Let us now consider the relation to type IIA string theory. For that, we assume that  $Y^{11}$  admits a differentiable circle action as in [8] [31] [42], and assume that the boundary  $\partial_i Y^{11}$  is invariant under this circle action. We would like to identify the corner in this case. The set of nonzero normal vectors  $\{v\}$  is  $N^{11} - W^6$ . This is acted upon by the positive real line  $\mathbb{R}_+^* = \{\alpha \in \mathbb{R} \mid \alpha > 0\}$  via multiplication by a positive scalar:  $v \mapsto \alpha v$ . The sphere bundle  $\mathcal{S}^{10}$  can be identified with the quotient  $(N - W^6)/\mathbb{R}_+^*$ . We extend to the cylinder bundle over the sphere bundle  $\mathcal{S}^{10} \times \mathbb{R} = \mathcal{C}^{11}$ , which is a trivial line bundle the fiber at  $\mathbb{R}_+^* v$  being  $\mathbb{R}v$ , and identify  $\mathcal{S}^{10}$  with the zero section of  $\mathcal{C}^{11}$ . Let  $\mathcal{C}_+^{11} \subset \mathcal{C}^{11}$  be the non-negative half of  $\mathcal{C}^{11}$ ; an element  $rv$  of the fiber of  $\mathcal{C}^{11}$  over  $\mathbb{R}_+^* v$  is in  $\mathcal{C}_+^{11}$  if  $r \geq 0$ . If  $U$  is an open neighborhood of  $W^6$  in  $N^{11}$ , denote by  $\mathcal{C}_+ U$  the inverse image of  $U$  under the canonical map  $\mathcal{C}_+^{11} \rightarrow N^{11}$ .

The  $S^1$ -manifold with corners of codimension-two will be, as a set, the disjoint union  $(Y^{11} - W^6) \cup \mathcal{S}^{10}$ . Define a tubular neighborhood map, that is an  $S^1$ -equivariant diffeomorphism  $T$  of an open  $S^1$ -invariant neighborhood  $U$  of  $W^6$  in  $N^{11}$  onto an open neighborhood  $U'$  of  $W^6$  in  $Y^{11}$  with the properties that  $T|_{W^6}$  is the inclusion map  $W^6 \subset Y^{11}$  and the induced map  $T_* : N^{11} \rightarrow N^{11}$  of the normal bundle of  $W^6$  in  $U$  into the normal bundle of  $W^6$  in  $U'$  is the identity map. The tubular map (see [9]) induces a map  $T' : \mathcal{C}_+ U \rightarrow (Y^{11} - W^6) \cup \mathcal{S}^{10}$ , with respect to which  $T'$  is a diffeomorphism onto a neighborhood of  $\mathcal{S}^{10}$  and which induces the given structure on  $(Y^{11} - W^6)$ . Now let  $T_1$  be a second tubular map, thus defining a second structure on  $(Y^{11} - W^6) \cup \mathcal{S}^{10}$ . Then the identity map on  $(Y^{11} - W^6) \cup \mathcal{S}^{10}$  is an isomorphism of the these two structures, so that  $(Y^{11} - W^6) \cup \mathcal{S}^{10}$  becomes a well-defined manifold with corners.

Let  $p : (Y^{11} - W^6) \cup \mathcal{S}^{10} \rightarrow Y^{11}$  be the natural projection, which is the identity on  $(Y^{11} - W^6)$  and bundle projection on  $\mathcal{S}^{10}$ . Define the boundary to be  $\partial_i((Y^{11} - W^6) \cup \mathcal{S}^{10}) = p^{-1}(\partial_i Y^{11})$  for  $i = 0, 1$

$$\begin{array}{ccc} (Y^{11} - W^6) \cup \mathcal{S}^{10} & \xleftarrow{\quad} & \partial_i((Y^{11} - W^6) \cup \mathcal{S}^{10}) \\ \downarrow \text{id} \times \text{pr} & & \downarrow p \\ Y^{11} & \xleftarrow{\quad} & \partial_i Y^{11} \end{array} \quad (2.6)$$

and  $\partial_2((Y^{11} - W^6) \cup \mathcal{S}^{10}) = \mathcal{S}^{10}$ . This shows that  $((Y^{11} - W^6) \cup \mathcal{S}^{10})$  becomes an  $S^1$ -manifold with corners of codimension-two. We summarize

**Proposition 3** *The M5-brane worldvolume in an eleven-dimensional manifold with boundary is a manifold with corners, described above.*



### 2.2.3 The M2-brane and boundaries

Consider M-theory on a Spin eleven-manifold  $Y^{11}$  which is a product of two Spin manifolds  $X^3 \times M^8$ . Take  $X^3$  to be a three-manifold with a boundary  $\partial X^3 = \Sigma_g$ , a Riemann surface, and  $M^8$  a closed eight-manifold. Now take an M2-brane with boundary to wrap around  $X^3$  and identify the boundary of the M2-brane with the boundary of  $X^3$ . Then we try to lift to twelve dimensions by making  $M^8$  into a boundary of a nine-dimensional manifold  $N^9$ , with  $\partial N^9 = M^8$ . However, we cannot always perform these steps because the Spin cobordism group in eight dimensions is not zero. In fact,  $\Omega_8^{\text{Spin}} \cong \mathbb{Z} \oplus \mathbb{Z}$ , generated by the quaternionic projective plane  $\mathbb{H}P^2$  and a generator which is one-fourth the square of the Kummer surface  $\frac{1}{4}(K3)^2$ . If we were to always find a Spin boundary then we would consider  $\Sigma_g \times M^8$  is the corner of the twelve-dimensional manifold  $Z^{12}$ . The latter is the product of two manifolds with boundary, namely  $X^3$  and  $N^9$ , and so indeed it is a manifold with corners of codimension-two.

We could also try to take  $Z^{12}$  to be just oriented and not necessarily Spin, and the same for  $M^8$ . Then in trying to lift from  $M^8$  to  $N^9$  (again just oriented) we have to check that the obstruction in the oriented cobordism group in dimension eight is zero. In general this is not the case since  $\Omega_8 \cong \mathbb{Z} \oplus \mathbb{Z}$ , generated by the projective spaces  $\mathbb{C}P^4$  and  $\mathbb{C}P^2 \times \mathbb{C}P^2$ . We then have

**Proposition 4** *The M2-brane with a boundary gives rise to a ten-dimensional corner in the twelve-dimensional theory, provided an eight-dimensional zero bordism is used for the transverse space.*

### 2.2.4 The heterotic theory as a corner

The topological and global analytic aspects of M-theory are best described using a lift from eleven dimensions to twelve dimensions, where the theory on  $Y^{11}$  is considered from the point of view of the theory on a twelve-dimensional Spin manifold  $Z^{12}$  bounding  $Y^{11}$ , that is  $\partial Z^{12} = Y^{11}$  [48]. On the other hand, heterotic string theory can be considered on M-theory with a boundary, that is when  $Y^{11}$  itself has a boundary. The naive boundary of a boundary does not exist. However, manifolds with corners come to the rescue, so that heterotic string theory can be viewed as a corner of the twelve-dimensional theory and the picture is consistent.

The rest of the paper will concerned with expanding around this interpretation. In the following section we will consider analytical and geometric consequences of viewing the heterotic theory as a corner.

## 3 Analytical and geometric aspects of M-theory with corners

The goal of this section is to explore analytical consequences of taking the heterotic theory to be a corner in the twelve-dimensional theory. Our discussion will mostly focus on the Dirac and signature operators, their eta-invariants, and the corresponding phase of the partition function.

### 3.1 Formulation using Dirac operators

In this section we consider M-theory on two disconnected components, both of which form the boundary of the twelve-dimensional bounding theory, and which intersect on one corner, representing the heterotic theory. This is the opposite to the usual situation, where M-theory is taken on one component and the heterotic theory is taken on two disconnected components. The analytical constructions we apply here are very nicely surveyed in [28], to which we refer heavily throughout this section. Mass regularizations and perturbations will play an important role.

**The fields in heterotic string theory.** Let  $S^\pm$  denote the Spin bundles on  $M^{10}$ . The fermionic fields in heterotic string theory consist of a gravitino  $\psi$ , which is a section of  $T^*M^{10} \otimes S^+$ , a dilatino  $\lambda$ , which is a  $\mathfrak{e}_8$ -valued section of  $S^+$ , and a gaugino  $\chi$ , which is a section of  $S^-$ . Here  $\mathfrak{e}_8$  is the Lie algebra of the Lie group  $E_8$ . We will work with general twisted spinors, that is with sections of  $S^\pm \otimes E$ , where the vector

bundle  $E$  can be taken as the  $E_8$  bundle or the tangent bundle (minus appropriate number of trivial line bundles) according to the context.

**The case with no corners.** For comparison, let us briefly recall the case with no corners [14]. Consider  $Y^{11} = [0, 1] \times M^{10}$  with the product metric, where  $M^{10}$  is a closed ten-dimensional Spin manifold. Then  $\partial Y^{11} = M_0 \cup M_1$ , where  $M_1 \cong M^{10}$  and  $M_0 \cong -M^{10}$  (that is,  $M^{10}$  with the opposite orientation). Let  $P^\pm$  be the local boundary conditions for the Dirac operator  $D_Y$  corresponding to spinors in  $S_{\partial Y}^\pm$  being zero, imposed respectively on  $M_0$  and  $M_1$ . Then [14]

$$\text{index}(D_Y, P^\pm) = \text{index}(D_M), \quad (3.1)$$

where  $D_M$  is the Dirac operator on  $M^{10}$ . As explained in [14], this is the case for Horava-Witten theory [20] [21], which we consider in more detail at the end of this section. The heterotic theory can also be viewed from the point of view of reduction of M-theory on  $S^1/\mathbb{Z}_2$ , where  $\mathbb{Z}_2$  is an orientation-reversing involution. More generally, let  $Y^{11}$  be an eleven-manifold with an orientation-reversing isometric involution  $\tau : Y^{11} \rightarrow Y^{11}$  and with a lift  $\tilde{\tau} : SY^{11} \rightarrow SY^{11}$  to the Spin bundle which anticommutes with the Dirac operator  $D_Y$  and satisfies  $\tilde{\tau}^2 = 1$ . Then  $D_Y : S^\pm Y^{11} \mapsto S^\mp Y^{11}$ , where  $S^\pm Y^{11}$  are the  $\pm$ -eigenspaces of  $\tilde{\tau}$ . When  $Y^{11} = S^1 \times M^{10}$  with  $\tau$  a reflection on the circle  $S^1$  and  $M^{10}$  is a compact Spin ten-manifold, then the same formula (3.1) holds [14].

**The case with corners.** Now consider  $Z^{12}$  as a compact oriented Riemannian 12-manifold with corners of codimension-two and metric  $g^Z$ . Assume that  $Z^{12}$  has exactly two boundary hypersurfaces  $Y_1^{11}$  and  $Y_2^{11}$  that intersect in exactly one codimension-two face  $M^{10}$ . The two hypersurfaces correspond to M-theory on two disconnected spaces. Near each hypersurface  $Y_i^{11}$ , we assume that  $Z^{12}$  has a collar neighborhood  $Z^{12} \cong [0, 1)_{s_i} \times Y_i^{11}$  where the metric is a product  $g^Z = ds_i^2 + g_i^Y$ , with  $g_i^Y$  the metric on  $Y_i^{11}$ . Then the product decomposition near each  $Y_i^{11}$  can be taken to be  $Z^{12} \cong [0, 1)_{s_1} \times [0, 1)_{s_2} \times M^{10}$  near the corner where the metric is a product  $g^Z = ds_1^2 + ds_2^2 + g^M$ , with  $g^M$  a metric on  $M^{10}$ . Here  $s_1$  and  $s_2$  are, as before, the coordinates on the ‘square’ over  $M^{10}$ . In what follows we apply some of the results (surveyed) in [28].

**The resulting Dirac operators on  $Y_i^{11}$  and on  $M^{10}$  starting from one on  $Z^{12}$ .** Let  $E$  and  $F$  be Hermitian vector bundles over  $Z^{12}$  whose restrictions to  $Y_i^{11}$  are  $E_i$  and  $F_i$ ,  $i = 1, 2$ , respectively. The restrictions to  $M^{10}$  are denoted  $E_0$  and  $F_0$ . What we have in mind are Spin bundles, possibly twisted by vector bundles, such as an  $E_8$  vector bundle or the tangent bundle. Starting with a Spin bundle  $S_Z = S_Z^+ \oplus S_Z^-$  on  $Z^{12}$ , this reduces to  $S_Y = S_Z^+$  or  $S_Y = S_Z^-$  on  $Y^{11}$ . The choice depends on the boundary conditions. In turn, the restriction of  $S_Y$  to the ten-dimensional boundary will be  $S_Y|_{M^{10}} \cong S_M \cong S_M^+ \oplus S_M^-$ . The two splittings lead to local boundary conditions for the Dirac operators on  $Z^{12}$  and on  $Y^{11}$ .

Let  $D : C^\infty(Z^{12}, E) \rightarrow C^\infty(Z^{12}, F)$  be a Dirac operator on  $Z^{12}$  which is of product type  $D = \Gamma_i(\partial_{s_i} + D_i)$  near each hypersurface on the collar  $Z^{12} \cong [0, 1)_{s_i} \times Y_i^{11}$ , where  $\Gamma_i$  is a Dirac matrix, i.e. a unitary isomorphism from  $E_i$  into  $F_i$  and where  $D_i : C^\infty(Y_i^{11}, E_i) \rightarrow C^\infty(Y_i^{11}, E_i)$  is a (formally) self-adjoint Dirac operator on the 11-dimensional manifold with boundary  $Y_i^{11}$ . Furthermore, assume that on the product decomposition near the corner, the Dirac operator takes the form  $D = \Gamma_1 \partial_{s_1} + \Gamma_2 \partial_{s_2} + B$  where  $B : C^\infty(M^{10}, E_0) \rightarrow C^\infty(M^{10}, F_0)$  is a Dirac operator on the ten-dimensional manifold without boundary  $M^{10}$ . On the collar  $Z^{12} \cong [0, 1)_{s_1} \times [0, 1)_{s_2} \times M^{10}$  we have  $\Gamma_i(\partial_{s_i} + D_i) = \Gamma_i \partial_{s_i} + B$ , ( $i = 1, 2$ ), so that

$$D_1 = \Gamma_1^{-1} \Gamma_2 \partial_{s_2} + \Gamma_1^{-1} B \quad \text{and} \quad D_2 = \Gamma_2^{-1} \Gamma_1 \partial_{s_1} + \Gamma_2^{-1} B \quad (3.2)$$

The fact that each  $D_i$  is (formally) self-adjoint,  $D_i^* = D_i$  is compatible with the Clifford algebra identity  $\Gamma_i^{-1} \Gamma_j + \Gamma_j^{-1} \Gamma_i = 2\delta_{ij}$  and gives the condition  $B^* \Gamma_i = \Gamma_i^{-1} B$  (here  $\Gamma^{-1} = \Gamma^*$ ). Then we relate the 11-dimensional operator to the 10-dimensional operator via  $D_1 = \Gamma(\partial_{s_2} + D_M)$ , where  $\Gamma = \Gamma_1^{-1} \Gamma_2$  and  $D_M = \Gamma_2^{-1} B$ . the operator  $D_M$  is the Dirac operator on  $M^{10}$  induced by  $D_1$ . The Dirac operator induced from  $D_2$  has a simple expression in relation to  $D_M$  and hence can be considered equivalent:  $D_2 = -\Gamma(\partial_{s_1} + \tilde{D}_M)$ , with  $\tilde{D}_M = \Gamma D_M$ .

Since  $\Gamma^2 = -\text{Id}$  then  $\Gamma : E_0 \rightarrow E_0$  has eigenvalues  $\pm i$ . Let  $E_0^\pm$  denote the eigenspaces corresponding to the eigenvalues  $\pm i$ . These are subbundles of  $E_0$  and

$$E_0 = E_0^+ \oplus E_0^- \quad (3.3)$$

is an orthogonal decomposition since  $\Gamma$  is unitary. Furthermore,  $D_M$  is odd with respect to  $\Gamma$ :  $D_M \Gamma = -\Gamma D_M$ , so  $D_M$  is also odd with respect to the  $\mathbb{Z}_2$ -grading (3.3). Therefore,

**Lemma 5** *The Dirac operator on the ten-dimensional heterotic corner  $M^{10}$  induced from twelve dimensions via  $D = \Gamma_1 \partial_{s_1} + \Gamma_2 \partial_{s_2} + \Gamma_2 D_M$  takes the form*

$$D_M = \begin{bmatrix} 0 & D_M^- \\ D_M^+ & 0 \end{bmatrix} : C^\infty(M^{10}, E_0^+ \oplus E_0^-) \rightarrow C^\infty(M^{10}, E_0^+ \oplus E_0^-), \quad (3.4)$$

where  $D_M^\pm$  are the restrictions of  $D_M$  to  $C^\infty(M^{10}, E_0^\pm)$ . Self-adjointness of  $D_M$  implies that  $(D_M^+)^* = D_M^-$ .

Here  $E_0^\pm$  is  $S_M^\pm$  twisted with the  $E_8$  vector bundle. Note that  $D_M^+$  is the operator appearing in Horava-Witten theory [20] [21].

**Square-integrability and Sobolev spaces.** We have seen in section 2 that manifolds with corners require working with square integrable differential forms. Denote the Sobolev space of order  $k$  by  $\mathcal{H}^k$ . Thus, for  $E$  a vector bundle as above,  $\mathcal{H}^k(M^{10}, E)$  denotes the  $E$ -spinors on  $M^{10}$  for which  $(D_M)^j \psi$ ,  $j = 0, \dots, k$ , is square-integrable. So  $\mathcal{H}^1(M^{10}, E)$  is the natural domain for  $D_M$ . From the Atiyah-Singer index theorem, the index of the Dirac operator

$$D_M^+ : \mathcal{H}^1(M^{10}, E_0^+) \rightarrow L^2(M^{10}, E_0^-) \quad (3.5)$$

is zero,  $\text{Ind} D_M^+ = 0$ , as in [14]. Since  $(D_M^+)^* = D_M^-$ , it follows that  $\dim \ker D_M^+ = \dim \ker D_M^-$ . Now consider the Dirac operator  $D : \mathcal{H}^1(Z^{12}, E) \rightarrow L^2(Z^{12}, F)$  on  $Z^{12}$ . This is never Fredholm as  $\dim \ker D = \infty$ . Therefore, we need to replace this operator by another (in a sense equivalent) Dirac operator which is Fredholm.

Let  $\widehat{Z}^{12}$  be the manifold formed by taking the infinite cylinder  $(-\infty, 0]_{s_1} \times Y_1^{11}$  and attaching it to the collar  $[0, 1)_{s_1} \times Y_1^{11}$  of  $Z^{12}$ , then taking  $(-\infty, 0]_{s_2} \times Y_2^{11}$  and attaching it to the collar  $[0, 1)_{s_2} \times Y_2^{11}$ , and finally taking  $(-\infty, 0]_{s_1} \times (-\infty, 0]_{s_2} \times M^{10}$  and attaching it to the remaining open quadrant. Since all geometric structures and the Dirac operator are of product type near the boundary of  $Z^{12}$ , they all have natural extensions to the manifold  $\widehat{Z}^{12}$ . Let  $\widehat{D}$  be the extension of the Dirac operator  $D$  to  $\widehat{Z}^{12}$ . When attaching ends to a manifold, one also talks about weighted Sobolev spaces which arise by considering the weighted (or conformal) Dirac operator  $e^{-\alpha s} \widehat{D} e^{\alpha s}$ , where  $s$  is a coordinate function on the cylindrical end and  $\alpha$  is a constant whose absolute value is less than the smallest absolute value of a nonzero eigenvalue of  $D_M$ . From  $e^{-\alpha s} \widehat{D} e^{\alpha s} = D + \alpha \Gamma$  we see that  $\alpha$  plays the role of mass for the spinors. Then  $e^{-\alpha s} \widehat{D} e^{\alpha s} : \mathcal{H}^1(\widehat{Z}^{12}, E) \rightarrow L^2(\widehat{Z}^{12}, F)$  can be replaced, for  $|\alpha| > 0$  sufficiently small, by

$$\widehat{D} : e^{\alpha s} \mathcal{H}^1(\widehat{Z}^{12}, E) \rightarrow e^{\alpha s} L^2(\widehat{Z}^{12}, F). \quad (3.6)$$

The variable  $s$  can be replaced with the variable  $x = e^s$ . We will shortly use two variables, one for each interval (see expression (3.7)). We summarize the conditions on the spinors

**Lemma 6** (i) *Both the spinor  $\psi$  and the mass-normalized spinor  $m\psi$  have to be integrable on  $\widehat{Z}^{12}$ .*  
(ii) *The normalized spinors on  $\widehat{Z}^{12}$  vanish as  $s \rightarrow \infty$  and coincide with the spinors when  $s = 0$ .*

Below we consider conditions, coming from the corner, for when  $\widehat{D}$  is Fredholm. We will consider two cases, according to whether or not the corner Dirac operator is invertible.

### 3.1.1 The non-supersymmetric case

In this section we assume that the corner Dirac operator  $D_M$  is invertible. Consider  $\widehat{Y}_i^{11}$ , the eleven-manifold with cylindrical end formed by attaching an infinite cylinder to the eleven-dimensional compact manifold with boundary  $Y_i^{11}$ . This has infinite volume so that the spectrum of the corresponding Dirac operator  $\widehat{D}_i$  is continuous rather than discrete. Then the eta-invariant cannot be defined since it involves a trace. As explained in section 2.1, the way around this is to use a  $b$ -trace.

Unlike the case of a manifold with boundary, on a manifold with corners the Dirac operator on  $Z^{12}$  cannot always be made into a Fredholm operator. In fact, [29] there exists a  $\delta > 0$  such that for all  $0 < |\alpha_i|, \delta$ ,  $i = 1, 2$ , the Dirac operator on weighted Sobolev spaces

$$\widehat{D} : e^{\alpha_1 s_1} e^{\alpha_2 s_2} \mathcal{H}^1(\widehat{Z}^{12}, E) \rightarrow e^{\alpha_1 s_1} e^{\alpha_2 s_2} L^2(\widehat{Z}^{12}, F) \quad (3.7)$$

is Fredholm if and only if the corner operator  $D_M : \mathcal{H}^1(M^{10}, E_0) \rightarrow L^2(M^{10}, E_0)$  is invertible (has zero kernel). Generally, a Dirac operator on a noncompact manifold is Fredholm if and only if it is invertible ‘at infinity’. That is [36] (also see [29]) the Dirac operator  $\widehat{D} : \mathcal{H}^1(\widehat{Z}^{12}, E) \rightarrow L^2(\widehat{Z}^{12}, F)$  is Fredholm if and only if  $\widehat{D}_i : \mathcal{H}^1(\widehat{Y}_i^{11}, E_i) \rightarrow L^2(\widehat{Y}_i^{11}, E_i)$  for  $i = 1, 2$ , and the corner operator  $D_M : \mathcal{H}^1(M^{10}, E_0) \rightarrow L^2(M^{10}, E_0)$  are each invertible. This places conditions on the topology of  $M^{10}$ .

Let us consider the index of the Dirac operator coupled to an  $E_8$  bundle  $E$  on the heterotic corner  $M^{10}$ . The integrand in the index is

$$\widehat{A}(M^{10}) \text{ch}(E) = c_1(E) \widehat{A}_2 + \text{ch}_3(E) \widehat{A}_1 + \text{ch}_5(E) . \quad (3.8)$$

This is automatically zero for  $E_8$  since in this case  $\text{ch}_i(E) = 0$ ,  $i = 1, 3, 5$ . Note that the characteristic classes of  $E$  are all in dimensions divisible by 4, and are given by  $248 + 60a + 6a^2 + \frac{1}{3}a^3$ , where  $a$  is the degree four class characterizing the bundle. Since  $\text{Index}(D_M) = \dim \ker D_M - \dim \text{coker } D_M$ , then the vanishing of the both the index and the dimension of the kernel gives that the cokernel is also trivial. Therefore, we have

**Proposition 7** *Requiring the Dirac operator in twelve dimensions to be Fredholm is equivalent to the Dirac operator on the heterotic corner being invertible, i.e. having zero kernel. This results in a non-supersymmetric theory. Furthermore, in this case the cokernel is also zero.*

An example of a non-supersymmetric heterotic theory is the model given in [11].

We have seen (cf. just before Lemma 6) that  $\widehat{Z}^{12}$  can be transformed to  $Z^{12}$ , with the metric transforming as in (2.5). Similarly, the Dirac operator transforms to the  $b$ -Dirac operator as

$$\widehat{D} = \Gamma_1 \partial_{s_1} + \Gamma_2 \partial_{s_2} + B \rightsquigarrow {}^b \widehat{D} = \Gamma_1 x_1 \partial_{x_1} + \Gamma_2 \partial_{x_2} + B , \quad (3.9)$$

which acts as  ${}^b \widehat{D} : x^\alpha \mathcal{H}^1(Z^{12}, E) \rightarrow x^\alpha L_b^2(Z^{12}, F)$ , where  $\mathcal{H}_b^1$  and  $L_b^2$  denote  $b$ -Sobolev spaces and space of square integrable functions using the  $b$ -integral. This operator is Fredholm if and only if the corner operator  $D_M$  is invertible (has zero kernel) [36].

Note that for general boundaries – even without corners – the eta-invariant  $\eta(\widehat{D}_i)$  should in general be replaced with the  $b$ -eta-invariant  ${}^b \eta(\widehat{D}_i)$  via replacing the trace  $\text{Tr}$  with the  $b$ -trace  ${}^b \text{Tr}$ . In this case, the APS index theorem in the setting above becomes [36]

$$\text{ind}_\alpha \widehat{D} = \int_{Z^{12}} \widehat{A}(Z^{12}) \text{ch}(E) - \frac{1}{2} \sum_{i=1,2} \left\{ {}^b \eta(\widehat{D}_i) + \text{sign} \alpha \cdot \dim \ker \widehat{D}_i \right\} , \quad (3.10)$$

where  $\alpha = (\alpha_1, \alpha_2)$ . Then we can reformulate the phase of the partition function using the  $b$ -eta-invariant by specifying the vector bundles  $E$  and  $F$  to the  $E_8$  bundle and to the Rarita-Schwinger bundle. Hence,

**Proposition 8** *The phase of the partition function for general boundary conditions for the eleven-dimensional boundary is  $\exp 2\pi i \left[ \frac{1}{4} {}^b \eta_{E_8} + \frac{1}{8} {}^b \eta_{RS} \right]$ .*

**Remark on the number of zero modes.** Note that expression (3.10) can be rewritten in terms of  $b$ -calculus by using more general boundary conditions than the one in the original Atiyah-Patodi-Singer treatment [2]; they are called augmented APS boundary conditions [17]. This way the number of zero modes, i.e. the dimensions of kernels of the Dirac operators, are absorbed into the index. The number of zero modes taken mod 2, that is the mod 2 index of the  $E_8$  Dirac operator in ten dimensions, plays a crucial and extensive role in the discussions in [8], which use the APS boundary conditions. Then we can see that when we use augmented boundary conditions, these zero mode terms are ‘absent’, and hence presumably cannot detect an anomaly. Therefore, we see that the use of more general boundary conditions via  $b$ -calculus seems to drastically simplify the discussion in [8].

**Extension to more corners.** The above results still hold if  $Z^{12}$  has more than one corner provided that each corner Dirac operator has zero index. This allows the construction of a separate perturbation for each corner [29]. The assumption can be removed by including a larger class of perturbations called ‘overblown’  $b$ -smoothing operators [28].

### 3.1.2 The supersymmetric case

We now consider the case when the corner Dirac operator is not invertible, that is the Dirac operator has a nonzero kernel and so there are zero modes for the spinors, as appropriate for a supersymmetric theory. Dropping the invertibility assumption on the corner Dirac operator  $D_M^+$  requires the use of perturbations (see [28] for a description of the formalism). For comparison with the boundary-only case, see [14]. The operator (3.5) has zero index, that is  $\dim \ker D_M^+ = \dim \ker D_M^-$ . The kernel is exactly the obstruction to  $\hat{D}$  being a Fredholm operator on weighted Sobolev spaces, so the perturbations are chosen to be isomorphisms on the kernel. Let  $T : \ker D_M \rightarrow \ker D_M$  be a self-adjoint unitary isomorphism that anticommutes with  $\Gamma = \Gamma_1^{-1}\Gamma_2$  (see expressions (3.2)), so  $T$  decomposes as an odd matrix

$$T = \begin{bmatrix} 0 & T^- \\ T^+ & 0 \end{bmatrix} : \ker D_M^+ \oplus \ker D_M^- \rightarrow \ker D_M^+ \oplus \ker D_M^-, \quad (3.11)$$

where  $T^\pm : \ker D_M^\pm \rightarrow \ker D_M^\mp$  are unitary isomorphisms with respect to the inner product on  $\ker D_M \subset L^2(M^{10}, E_0)$ .

Let  $\{\psi_j^+\}_{j=1}^N$  and  $\{\psi_j^-\}_{j=1}^N$  be spinor orthonormal bases of  $\ker D_M^+$  and  $\ker D_M^-$ , respectively, with  $\psi_j^+, \psi_j^- \in C^\infty(M^{10}, E_0)$  spinors on  $M^{10}$ , possibly twisted with the tangent bundle or with the  $E_8$  vector bundle. Then  $T$  has the expression in terms of fermion bilinears

$$T = \sum_{j=1}^N \psi_j^+ \otimes \bar{\psi}_j^- + \sum_{j=1}^N \psi_j^- \otimes \bar{\psi}_j^+. \quad (3.12)$$

and is a smoothing operator on  $M^{10}$ , that is  $T$  has a smooth kernel. Since  $\ker D_M^+$  is a dual vector space to  $\ker D_M^-$ , we interpret  $T$  as a mass operator giving rise to a mass term. Then the massive operator  $D_M - T : \mathcal{H}^1(M^{10}, E_0) \rightarrow L^2(M^{10}, E_0)$  is invertible.

Note that a mass term is needed to describe the contribution of the Rarita-Schwinger field to the partition function of M-theory on  $Y^{11}$  [48]. Consider the massive Rarita-Schwinger operator  $D_{RS}^m = D_{RS} + im$ , where  $m$  is a constant so that  $im$  is a soft perturbation. Different limits are obtained for  $m \rightarrow \pm\infty$ , so that the determinant is  $\det D_{RS} \exp(\pm i I_{RS})$ , with the sign depending on the sign of  $m$ . Here  $I_{RS}$  is the Rarita-Schwinger index in twelve dimensions. Therefore, this makes it only natural to work with massive Dirac operators in twelve dimensions.

In what follows we aim to characterize the effect of the corner on the phase of the partition function. The matrix  $T$  squares to the identity matrix  $T^2 = \text{Id}$ , so that  $T$  has eigenvalues  $\pm 1$ . Let  $\Lambda_T \subset \ker D_M$  be

the  $+1$ -eigenspace of  $T$  and let  $\Lambda_{\Gamma T} \subset \ker D_M$  is the  $+1$ -eigenspace of the self-adjoint unitary automorphism  $\Gamma T$ . That is,

$$\Lambda_T := \{\Psi \in \ker D_M : T\Psi = +\Psi\}, \quad \Lambda_{\Gamma T} := \{\Psi \in \ker D_M : \Gamma T\Psi = +\Psi\}. \quad (3.13)$$

Now, considering an extension  $\widehat{T}$  of  $T$  to  $Z^{12}$ , the operator

$$\widehat{D} - \widehat{T} : x^\alpha \mathcal{H}_b^1(Z^{12}, E) \rightarrow x^\alpha L_b^2(Z^{12}, F) \quad (3.14)$$

is Fredholm for all  $0 < |\alpha| < \delta$  for some  $\delta > 0$ . Recall that  $\widehat{Y}_1^{11}$  is formed by attaching an infinite cylinder  $(-\infty, 0]_{s_2} \times M^{10}$  to the eleven-dimensional compact manifold with boundary  $Y_1^{11}$ , and similarly for  $\widehat{Y}_2^{11}$ . Let  $\widehat{T}_1$  and  $\widehat{T}_2$  denote the operators induced by  $\widehat{T}$  on  $Y_1^{11}$  and  $Y_2^{11}$ , respectively. We need to consider boundary conditions on the spinors at the infinite ends of the cylinders. The two sets

$$\begin{aligned} \Lambda_{C_1} &= \left\{ \lim_{s_2 \rightarrow -\infty} \Psi(s_2, y) : \Psi \in C^\infty(\widehat{Y}_1^{11}, E) \text{ is bounded, and } \widehat{D}_1 \Psi = 0 \right\}, \\ \Lambda_{C_2} &= \left\{ \lim_{s_1 \rightarrow -\infty} \Psi(s_1, y) : \Psi \in C^\infty(\widehat{Y}_2^{11}, E) \text{ is bounded, and } \widehat{D}_2 \Psi = 0 \right\}, \end{aligned}$$

are called the scattering Lagrangian subspaces of  $\widehat{D}_1$  and  $\widehat{D}_2$ , respectively. It turns out that, for each  $i = 1, 2$ ,  $\Lambda_{C_i} \subset \ker D_M$  and the dimension of  $\Lambda_{C_i}$  is exactly one-half the dimension of  $\ker D_M$  [33]. The scattering matrix of  $\widehat{D}_i$  is the operator  $C_i : \ker D_M \rightarrow \ker D_M$  defined by  $C_i = +1$  on  $\Lambda_{C_i}$  and  $C_i = -1$  on  $\Lambda_{C_i}^\perp$ , where “ $\perp$ ” stands for orthogonal complement with respect to the  $L^2$  inner product. Then  $C_1$  and  $C_2$  are odd with respect to  $\Gamma$  (see [35]).

**Effect of the interval and the mass.** The  $b$ -eta-invariants and the dimensions of the kernels of the massive operators on  $Y_i^{11}$  ( $i = 1, 2$ ) can be given in terms of their massless counterparts as

$$\begin{aligned} \dim \ker(\widehat{D}_i - \widehat{T}_i) &= \dim \ker \widehat{D}_i + \dim(\Lambda_{T_i} \cap \Lambda_{C_i}), \\ \mathbf{b}_\eta(\widehat{D}_i - \widehat{T}_i) &= \mathbf{b}_\eta(\widehat{D}_i) \pm \mu(\Lambda_{T_i}, \Lambda_{C_i}), \end{aligned}$$

where  $T_1 = T$  and  $T_2 = \Gamma T$ , and the upper and lower signs are taken for  $i = 1$  and  $2$ , respectively. Here  $\mu(\Lambda_T, \Lambda_{C_1})$  and  $\mu(\Lambda_{\Gamma T}, \Lambda_{C_2})$  are spectral expressions [27] which can be interpreted as an ‘exterior angle’ between the Lagrangian subspaces [3]. Then the index in this case is given in terms of the usual (but with  $b$ -calculus) APS terms plus a correction due to the corner, namely [29]

$$\begin{aligned} \text{ind}_\alpha(\widehat{D} - \widehat{T}) &= \int_{Z^{12}} \widehat{A}(Z^{12}) - \frac{1}{2} \sum_{i=1}^2 \left\{ \mathbf{b}_\eta(\widehat{D}_i - \widehat{T}_i) + \text{sign} \alpha \cdot \dim \ker(\widehat{D}_i - \widehat{T}_i) \right\} \\ &= \int_{Z^{12}} \widehat{A}(Z^{12}) - \frac{1}{2} \sum_{i=1}^2 \left\{ \mathbf{b}_\eta(\widehat{D}_i) \pm \mu(\Lambda_{T_i}, \Lambda_{C_i}) + \text{sign} \alpha \cdot \left( \dim \ker(\widehat{D}_i) + \dim(\Lambda_{T_i} \cap \Lambda_{C_i}) \right) \right\} \\ &= \int_{Z^{12}} \widehat{A}(Z^{12}) - \frac{1}{2} \sum_{i=1}^2 \left\{ \mathbf{b}_\eta(\widehat{D}_i) + \text{sign} \alpha \cdot \dim \ker \widehat{D}_i \right\} - \frac{1}{2} c_\alpha(\Lambda_T, \Lambda_{C_1}, \Lambda_{C_2}). \end{aligned}$$

The correction term is given by the expression

$$c_\alpha(\Lambda_T, \Lambda_{C_1}, \Lambda_{C_2}) = \dim(\Lambda_T \cap \Lambda_{C_1}) + \mu(\Lambda_T, \Lambda_{C_1}) + \dim(\Lambda_{\Gamma T} \cap \Lambda_{C_2}) - \mu(\Lambda_{\Gamma T}, \Lambda_{C_2}). \quad (3.15)$$

The precise value of this term will require explicit evaluation for a given situation. Generally, we then have

**Proposition 9** *The correction to the phase due to the presence of a corner, for the case when the Dirac operator on the ten-dimensional corner is not invertible, is given  $\exp \pi i$  times the term  $c_\alpha(\Lambda_T, \Lambda_{C_1}, \Lambda_{C_2})$ .*

We will see below in section 3.2 (in terms of the signature) that such a term will be zero in many simple cases of interest.

## 3.2 Formulation using the signature operator

In [44] we formulated the phase of the partition function by using the signature operator  $\mathcal{S}$  and the signature  $\sigma(Z^{12})$  of the twelve-manifold  $Z^{12}$ . That was done for the case when M-theory is taken on an eleven-manifold without boundary. In this section we will consider the extension to the case when there are corners present. In section 3.2.1 we consider the case when the relation between the ten-dimensional manifold  $M^{10}$  and the twelve-dimensional manifold  $Z^{12}$  is through a ‘fiber’ that is a product of two intervals. Then in section 3.2.2 we formulate the problem using orientation-reversing involutions. The first is related to the formulation of heterotic string theory as a boundary in M-theory, and the second is related to viewing that theory as the base of a circle bundle with an orientation-reversing involution on the fiber. This is a global extension of the [20] [21] “upstairs” and “downstairs” notions, respectively, to corners.

### 3.2.1 The phase of the partition function for manifolds with corners via the signature

We begin by recalling some properties of the signature that will be useful for us. For  $Z^{12}$  closed and oriented, Hodge theory implies that the index of the signature operator  $\mathcal{S}$  is given as the integral  $\sigma(Z^{12}) = \text{index}(\mathcal{S}) = \int_{Z^{12}} L$ , where  $L$  is the Hirzebruch L-polynomial and the right-hand side is the signature of the quadratic form on  $H^6(Z^{12}; \mathbb{R})$  given by the cup product [19]. There is a bilinear form on  $H^6(Z^{12}) \otimes H^6(Z^{12}) \xrightarrow{\cup} \mathbb{R}$ , where the cup product  $\cup$  is symmetric and nondegenerate, and the signature of  $Z^{12}$  is  $\sigma(Z^{12}) = \sigma(\cup)$  with the following properties

1. *Orientation reversal:*  $\sigma(-Z^{12}) = -\sigma(Z^{12})$ . This will be relevant when we connect to heterotic string theory via the Horava-Witten involution [20] [21].
2. *Product:*  $\sigma(M^4 \times N^8) = \sigma(M^4)\sigma(N^8)$ . This will be useful in compactifications to four dimensions and to relating the corresponding secondary invariants in eleven dimensions to those in seven dimensions.
3. *Bordism invariance:* If  $Z^{12} = \partial W^{13}$  then  $\sigma(Z^{12}) = 0$ . In this case the deficit, that is the  $\eta$ -invariant, will be given by the L-genus.
4. *Novikov Additivity:* For  $Z^{12} = Z_1^{12} \cup_{\partial Z_1^{12} = \partial Z_2^{12}} Z_2^{12}$ , the relation  $\sigma(Z^{12}) = \sigma(Z_1^{12}) + \sigma(Z_2^{12})$  holds [37]. We will use a variation on this, that is Wall’s non-additivity [47]; see (3.19) below. The common boundary will be (components of) the eleven-manifold on which M-theory is studied.

Consider the case when M-theory is taken on an eleven-manifold  $Y^{11}$  with boundary  $\partial Y^{11} = M^{10}$ . The partition function in this case is studied in [7], where also multiple boundaries  $\partial Y^{11} = \bigcup_i M_i^{10}$  were allowed. Now take the bounding twelve-manifold  $Z^{12}$  to be partitioned into two manifolds  $Z_1^{12}$  and  $Z_2^{12}$ , which have boundaries, which in turn have ‘boundaries’, so that the twelve-manifold  $Z^{12}$  is a manifold with corners of codimension-two, with the corners being the union of ten-manifolds  $M_i^{10}$ . In this case we use the index theorems for manifolds with corners, as constructed in [3] [36] [17]. What replaces a two-disk  $\mathbb{D}^2$  for the fiber over the ten-manifold is topologically a square, that is topologically a product of two intervals  $I \times I$ . For simplicity, we will consider unit intervals for the rest of this section.

We will next consider the corresponding eta-invariants.

**Signature eta-invariant for corners.** Consider  $Z^{12}$  to have a boundary  $Y^{11}$  which has a neighborhood metrically of the form  $Y^{11} \times [0, 1)$ . Consider  $M^{10} \subset Y^{11}$  as a separating closed ten-dimensional submanifold which possesses a neighborhood metrically of the form  $M^{10} \times (-1, 1)$ . In this setting, the corresponding eta-invariant can be defined as the signature defect [40]

$$\eta(Y^{11}, M^{10}) = \int_{Z^{12}} L_{12} - \sigma(Z^{12}). \quad (3.16)$$

The ten-manifold  $M^{10}$  is a corner of  $Z^{12}$ , that is,  $\partial Z^{12} = Y^{11}$  with a neighborhood of  $Y^{11} - M^{10}$  of the form  $(Y^{11} - M^{10}) \times [0, 1)$  and a neighborhood of  $M^{10}$  metrically of the form  $M^{10} \times ([0, 1) \times [0, 1))$ . For

two twelve-manifolds  $Z_1^{12}$  and  $Z_2^{12}$  having  $Y^{11}$  as a common boundary and  $M^{10}$  as a common corner, the invariant  $\eta(Y^{11}, M^{10})$  is well-defined [40], that is takes the same value on both twelve-manifolds,

$$\int_{Z_1^{12}} L_{12} - \sigma(Z_1^{12}) = \int_{Z_2^{12}} L_{12} - \sigma(Z_2^{12}) . \quad (3.17)$$

Furthermore, the above  $\eta$ -invariant of the pair  $(Y^{11}, M^{10})$  can be related to the eta-invariant of closed  $Y^{11}$  as follows (see [40]). Let  $Y'^{11} = \partial Z^{12}$  be a closed eleven-manifold and  $M^{10} \in Y'^{11}$  a closed separating submanifold as above. Assume there exists  $Y_0^{11} \subset Z^{12}$ , an eleven-dimensional submanifold with boundary  $\partial Y_0^{11} = M^{10} = \partial Z^{12} \cap Y_0^{11}$ , separating  $Z^{12}$  into two components  $Z_+^{12}$  and  $Z_-^{12}$  such that the neighborhoods of  $M^{10}$  in  $Z_{\pm}^{12}$  are metrically of the form  $M^{10} \times [0, 1) \times [0, 1)$  and neighborhoods of  $Y_0^{11}$  in  $Z_{\pm}^{12}$  metrically of the form  $Y_0^{11} \times [0, 1)$ . Let  $Y_{\pm}^{11}$  be the component of  $Y^{11} - M^{10}$  lying in  $Z_{\pm}^{12}$ , so that  $(Y_{\pm}^{11} \cup_M Y_0^{11} = N_{\pm}^{11}, M^{10})$  defines a pair of manifolds as above for each of  $\pm$ . Then the eta-invariants are related as follows

$$\eta(Y^{11}) = \eta(Y_+^{11}, M^{10}) - \eta(Y_-^{11}, M^{10}) + \delta(M^{10}; Y_-^{11}, Y_0^{11}, Y_+^{11}) , \quad (3.18)$$

where  $\delta$  is Wall's invariant, that is the obstruction to additivity of the signature [47]

$$\sigma(Z_+^{12} \cup_{Y_0^{11}} (-Z_-^{12})) = \sigma(Z_-^{12}) - \sigma(Z_+^{12}) + \delta(M^{10}; Y_-^{11}, Y_0^{11}, Y_+^{11}) . \quad (3.19)$$

Therefore, we see that we can 'trade' the boundary in M-theory with Wall's invariant. However, this invariant involves the homology groups  $H_5(M^{10})$  and  $H_5(Y^{11})$ , or dually the cohomology groups  $H^5(M^{10})$  and  $H^6(Y^{11})$ ; since M-theory does not support fields in these degrees, this results in the Wall invariant being zero in this case. Then the eta-invariant of the eleven-manifolds with boundary can be expressed in terms of the eta-invariants of the closed eleven-manifold, that is, (3.18) reduces to

$$\eta(Y^{11}) = \eta(Y_+^{11}, M^{10}) - \eta(Y_-^{11}, M^{10}) . \quad (3.20)$$

Note that the phase factor of the partition function in the presence of a boundary has the same expression as when there is no boundary, except of course that the terms would be a modification for the meaning of the terms [7] [15]. Therefore, the above result might not be surprising.

**Proposition 10** *The correction term to the phase in the signature formulation is given by Wall's non-additivity term. When  $H^5(M^{10})$  and  $H^6(Y^{11})$  are zero, then there is no correction.*

**Multiple boundary components.** Let  $Y_i^{11}$ ,  $i = 1, \dots, N$  be an ordering of codimension-one boundary components of  $Z^{12}$ . The intersections  $M_{ij}^{10} := Y_i^{11} \cap Y_j^{11}$ , which need not be connected, are the codimension-two boundaries, i.e. the corners. Let  $D_{ij}$  denote the Dirac operator on  $M_{ij}^{10}$  induced by either  $D_i$  on  $Y_i^{11}$  or  $D_j$  on  $Y_j^{11}$ . The two operators are the same up to sign and are induced by the signature operator on  $Z^{12}$ . Note that the signature itself can be viewed as a generalized Dirac operator. Assuming the APS boundary conditions, the signature can be written as [17]

$$\sigma(Z^{12}) = \int_{Z^{12}} L_{12} - \frac{1}{2} \left( \sum_{i=1}^N \eta(D_i) + \frac{1}{i\pi} \text{tr} P_{\Lambda} \right) , \quad (3.21)$$

where  $P_{\Lambda}$  is the corner analytical correction term involving a certain projection matrix  $P_{\Lambda}$  (see [17] for details). This is, in a sense, an analog of the correction term (3.15) in the Dirac operator case. We will not use the explicit form of this correction term in this paper, and below we consider examples where this is actually zero. Note that the number of zero modes does not appear in (3.21) since they are absorbed in the index, since this is defined with respect to a modified projection, called the augmented APS projection [17] (see also the remark right after proposition 8).



**Example 5. Products.** Consider  $Z^{12}$  as the product of two manifolds with boundary, so that  $Z^{12}$  is a manifold with corners of codimension-two. In this case  $N = 2$ . Let  $X^4$  and  $U^8$  be two manifolds with product  $b$ -metrics  ${}^b g_X$  and  ${}^b g_U$  and boundaries  $\partial X^4 = Y^3$  and  $\partial U^8 = V^7$ , respectively. The signatures of  $X^4$  and  $U^8$  are given by

$$\sigma(X^4) = \int_{X^4} L_4 - \frac{1}{2}\eta(Y^3), \quad \sigma(U^8) = \int_{U^8} L_8 - \frac{1}{2}\eta(V^7). \quad (3.22)$$

The signature of  $X^4 \times U^8$  is the product of signatures of the factors

$$\begin{aligned} \sigma(X^4 \times U^8) &= \int_{X^4 \times U^8} L_3 - \frac{1}{2}\eta(Y^3) \int_{U^8} L_8 - \frac{1}{2}\eta(V^7) \int_{X^4} L_2 + \frac{1}{4}\eta(Y^3)\eta(V^7) \\ &= \int_{X^4 \times U^8} L_{12} - \frac{1}{2}\eta(Y^3)\sigma(U^8) - \frac{1}{2}\eta(V^7)\sigma(X^4) - \frac{1}{4}\eta(Y^3)\eta(V^7). \end{aligned}$$

The  $b$ -eta-invariants are given by

$${}^b\eta(X^4 \times V^7) + {}^b\eta(Y^3 \times U^8) = \sigma(X^4) {}^b\eta(V^7) + \sigma(U^8) {}^b\eta(Y^3) + \eta(Y^3)\eta(V^7) \quad (3.23)$$

and account for the eta-terms in (3.21). The corner is  $M^{10} = Y^3 \times V^7 = \partial(X^4 \times V^7) - \partial(Y^3 \times U^8)$ . Identify the Lagrangians  $\Lambda_{12}$  and  $\Lambda_{21}$ , which are the asymptotic limit of solutions of the generalized Dirac equation  $D\psi = 0$  on  $X^4 \times V^7$  and  $Y^3 \times U^8$ , respectively. Using  $K_1 = \ker(D_Y)$ ,  $K_2 = \ker(D_V)$  and  $K = K_1 \otimes K_2 = \ker(D_{Y \times V})$ , and  $\Lambda_i \subset K_i$  denote the scattering Lagrangian for  $D_Y$  and  $D_V$ , respectively. A calculation using heat kernels shows that the corner correction term vanishes in this case (see [17]).

**Example 6. Disks.** Let us consider a very simple, but physically realistic case, where  $X^4 = \mathbb{D}^4$  and  $U^8 = \mathbb{D}^8$ , the 4- and 8-dimensional disks, respectively. Assume then that the corresponding boundaries  $Y^3 = S^3$  and  $V^7 = S^7$ , the round 3- and 7-spheres. Then in this case the eta-invariants vanish and so the signature formula for manifolds with corners reduces drastically all the way to that of a manifold without boundary, that is  $\sigma(\mathbb{D}^4 \times \mathbb{D}^8) = \int_{\mathbb{D}^4} L_1 \int_{\mathbb{D}^8} L_2$ . But then these are given in terms of Pontrjagin classes, which are zero in cohomology for disks.

**Proposition 11** *The phase of the partition function in the case of a product  $Z^{12} = X^4 \times U^8$  of two manifolds with boundary  $X^4$  and  $U^8$  is given by the Atiyah-Patodi-Singer index; that is, there is no corner correction in this case.*

### 3.2.2 Orientation-reversing involutions

In this section we consider the “downstairs” formulation of the heterotic theory, that is the case when  $Y^{11} = M^{10} \times S^1/\mathbb{Z}_2$  with an orientation-reversing involution  $\pi$  on  $S^1$  generated by  $\mathbb{Z}_2$ . We will in fact work in more generality in what follows.

Let  $Z^{12}$  be a compact oriented smooth twelve-manifold with boundary,  $Y^{11}$ . This boundary is called a *reflecting boundary* of  $Z^{12}$  if it admits an orientation-reversing involution  $\pi$ . A simple example of a reflecting boundary of  $Z^{12}$  is an eleven-sphere  $S^{11}$ . The *doubling* of the manifold  $Z^{12}$  with a reflecting boundary  $(Y^{11}, \pi)$  is a  $C^\infty$ -homeomorphism  $h : Z^{12} \rightarrow N^{12}$  where  $N^{12}$  is a smooth closed manifold with an involution  $\nu : N^{12} \rightarrow N^{12}$  such that we have an equality of compositions  $\nu \circ h \circ \iota = h \circ \iota \circ \pi : Y^{11} \rightarrow N^{12}$  in the diagram

$$\begin{array}{ccccc} N^{12} & \xleftarrow{h} & Z^{12} & \xleftarrow{\iota} & Y^{11} \\ \downarrow \nu & & \downarrow = & & \downarrow \pi \\ N^{12} & \xleftarrow{h} & Z^{12} & \xleftarrow{\iota} & Y^{11} \end{array} \quad (3.24)$$

A *symmetric* metric on the double  $N^{12}$  is a Riemannian metric for which the involution  $\nu$  is an isometry. Starting from a smooth Riemannian metric  $g$  on  $N^{12}$  one gets a smooth symmetric metric by setting, for  $x \in N^{12}$ , (see [22])

$$g(x) = \frac{1}{2} [g(x) + g(\nu(x))] . \quad (3.25)$$

Let the curvature of  $g$  be  $R_g$  and consider the corresponding Pontrjagin forms  $p_i(g)$ ,  $i = 1, 2, 3$ . The Hirzebruch signature theorem then gives

$$\sigma(N^{12}) = \int_{N^{12}} L_{12}(p_1(g), p_2(g), p_3(g)) . \quad (3.26)$$

The symmetric metric  $g$  on  $N^{12}$  restricts to a metric  $g_Z$  on  $Z^{12}$ , also called a symmetric metric. Then, applying [22], the signature of  $Z^{12}$  with a reflecting boundary  $Y^{11}$  is given by

$$\sigma(Z^{12}, Y^{11}) = \int_{Z^{12}} L_{12}(p_1(g_Z), p_2(g_Z), p_3(g_Z)) . \quad (3.27)$$

This reduces to the Hirzebruch signature theorem when  $Y^{11}$  is empty. Therefore, in the presence of a reflecting boundary, the Chern-Simons and one-loop terms are encoded in primary characteristic classes and the eta-invariant drops out.

**Proposition 12** *The signature part of the phase of the partition function is 1 when  $Y^{11}$  has a reflecting boundary.*

It would be interesting to work out examples where the correction terms are evaluated explicitly. We also plan to extend the discussion in this paper to more refined invariants.

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